

SOME BASIC PROPERTIES OF BLOCK OPERATOR MATRICES

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ABSTRACT. General approach to the multiplication or adjoint operation of 2×2 block operator matrices with unbounded entries are founded. Furthermore, criteria for self-adjointness of block operator matrices based on their entry operators are established.

1. INTRODUCTION

Block operator matrices arise in various areas of mathematical physics such as ordinary differential equations [22, 23, 26], theory of elasticity [15, 33, 34], hydrodynamics [10, 11], magnetohydrodynamics [14], quantum mechanics [28], and optimal control [13, 32]. The spectral properties of block operator matrices are of vital importance as they govern for instance the solvability and stability of the underlying physical systems. As a basis for spectral analysis, the multiplication or adjoint operation and self-adjointness of block operator matrices with unbounded entries have attracted considerable attention and have been investigated case by case, see, e.g., [3, 4, 7, 18, 19] for the former and [5, 6, 12, 17, 25] for the latter, or the monograph [29] for both of these topics. As has been pointed out in [18], what is essentially trivial for bounded operators appears to become erratic for unbounded operators. The purpose of this paper is to build a common framework for these problems.

To this end, let us first recall some notions on block operator matrices. Throughout this paper, we will denote by X_1, X_2 complex Banach spaces, X_1^*, X_2^* the adjoint spaces (see [9, Section III.1.4]), and $X := X_1 \times X_2$ the product space equipped with the norm

$$\|(x_1 \ x_2)^t\| := (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}.$$

It is well known that $(X_1 \times X_2)^*$ is isometrically isomorphic to $X_1^* \times X_2^*$ equipped with the norm

$$\|(f_1 \ f_2)^t\| := (\|f_1\|^2 + \|f_2\|^2)^{\frac{1}{2}}$$

such that if the element f of $(X_1 \times X_2)^*$ is identified with the element $(f_1 \ f_2)^t$ of $X_1^* \times X_2^*$, then

$$(f, x) = (f_1, x_1) + (f_2, x_2)$$

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whenever $x = (x_1 \ x_2)^t \in X_1 \times X_2$ (see [16, Theorem 1.10.13]). Following Engel [4], we define the injections J_1, J_2 and the projections P_1, P_2 as follows.

$$\begin{aligned} J_1 : X_1 \rightarrow X, \ J_1 x_1 &:= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \text{ and } P_1 : X \rightarrow X_1, \ P_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := x_1, \\ J_2 : X_2 \rightarrow X, \ J_2 x_2 &:= \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \text{ and } P_2 : X \rightarrow X_2, \ P_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := x_2. \end{aligned}$$

Furthermore, we denote by Q_1, Q_2 the projections

$$\begin{aligned} Q_1 : X \rightarrow X, \ Q_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &:= \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \\ Q_2 : X \rightarrow X, \ Q_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &:= \begin{pmatrix} 0 \\ x_2 \end{pmatrix}. \end{aligned}$$

Definition 1.1. ([31, p. 97]) Let $A_{jk} : \mathcal{D}(A_{jk}) \subset X_k \rightarrow X_j$ be linear operators, $j, k = 1, 2$. The matrix

$$(1.1) \quad \mathcal{A} := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is called a (2×2) block operator matrix on X . It induces a linear operator on X which is also denoted by \mathcal{A} :

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &:= (\mathcal{D}(A_{11}) \cap \mathcal{D}(A_{21})) \times (\mathcal{D}(A_{12}) \cap \mathcal{D}(A_{22})), \\ \mathcal{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &:= \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \end{aligned}$$

2. PRODUCT AND ADJOINT

In this section, we shall establish rules for the product and adjoint operations of block operator matrices.

Lemma 2.1. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \rightarrow X$ be a linear operator. The following statements are equivalent.

- (a) \mathcal{A} has a matrix representation (1.1),
- (b) $\mathcal{D}(\mathcal{A}) = P_1 \mathcal{D}(\mathcal{A}) \times P_2 \mathcal{D}(\mathcal{A})$,
- (c) $Q_1 \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ or, equivalently, $Q_2 \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$.

Furthermore, if one of the above statements holds, then

$$(2.1) \quad \mathcal{A} = \begin{pmatrix} P_1 \mathcal{A} J_1 & P_1 \mathcal{A} J_2 \\ P_2 \mathcal{A} J_1 & P_2 \mathcal{A} J_2 \end{pmatrix}$$

in the sense of linear operators on X .

Proof. First, the statements (a) and (c) are equivalent (see [27, p. 287]) and, moreover, one see easily that the statements (b) and (c) are equivalent. In addition, one readily checks that (2.1) holds if \mathcal{A} has a matrix representation, see also [4]. ■

Definition 2.1. Let $\mathcal{A} = (A_{jk}), \mathcal{B} = (B_{jk})$ be block operator matrices on X . We define the formal product block operator matrix of \mathcal{A} and \mathcal{B} as follows:

$$\mathcal{A} \times \mathcal{B} := \left(\sum_{k=1}^2 A_{jk} B_{kl} \right).$$

Theorem 2.1. *Let \mathcal{A}, \mathcal{B} be block operator matrices on X . Then $\mathcal{A} \times \mathcal{B} = \mathcal{AB}$ if and only if \mathcal{AB} has a matrix representation.*

Proof. The “only if” part is trivial.

The proof of the “if” part. Assume \mathcal{AB} has a matrix representation. Writing $\mathcal{A} = (A_{jk}), \mathcal{B} = (B_{jk})$ and $\mathcal{D}(\mathcal{A}) = \mathcal{D}_1 \times \mathcal{D}_2$, where \mathcal{D}_k are subspaces of X_k for $k = 1, 2$, respectively. Then

$$\begin{aligned}\mathcal{D}(\mathcal{A} \times \mathcal{B}) &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathcal{B}) \mid B_{11}x_1, B_{12}x_2 \in \mathcal{D}_1, B_{21}x_1, B_{22}x_2 \in \mathcal{D}_2 \right\}, \\ \mathcal{D}(\mathcal{AB}) &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathcal{B}) \mid B_{11}x_1 + B_{12}x_2 \in \mathcal{D}_1, B_{21}x_1 + B_{22}x_2 \in \mathcal{D}_2 \right\},\end{aligned}$$

and so $\mathcal{D}(\mathcal{A} \times \mathcal{B}) \subset \mathcal{D}(\mathcal{AB})$. One readily checks that $(\mathcal{A} \times \mathcal{B})x = \mathcal{AB}x$ for all $x \in \mathcal{D}(\mathcal{A} \times \mathcal{B})$ which implies $\mathcal{A} \times \mathcal{B} \subset \mathcal{AB}$. It remains to show $\mathcal{D}(\mathcal{AB}) \subset \mathcal{D}(\mathcal{A} \times \mathcal{B})$. If

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathcal{AB}),$$

then by Lemma 2.1,

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathcal{AB}),$$

and so from the structure of the set $\mathcal{D}(\mathcal{AB})$ we know that

$$\begin{aligned}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} &\in \mathcal{D}(\mathcal{B}), B_{11}x_1 \in \mathcal{D}_1, B_{21}x_1 \in \mathcal{D}_2, \\ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} &\in \mathcal{D}(\mathcal{B}), B_{12}x_2 \in \mathcal{D}_1, B_{22}x_2 \in \mathcal{D}_2,\end{aligned}$$

which imply $x \in \mathcal{D}(\mathcal{A} \times \mathcal{B})$ by the structure of the set $\mathcal{D}(\mathcal{A} \times \mathcal{B})$. Hence $\mathcal{D}(\mathcal{AB}) \subset \mathcal{D}(\mathcal{A} \times \mathcal{B})$. ■

The case \mathcal{AB} has no matrix representation can occur.

It follows from Theorem 2.1 the *Frobenius-Schur factorization* of a block operator matrix.

Corollary 2.1. ([29, Section 2.2]) *Let*

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a block operator matrix acting on X .

- (a) *Suppose that D is closed with $\rho(D) \neq \emptyset$, and that $\mathcal{D}(D) \subset \mathcal{D}(B)$. Then for some (and hence for all) $\lambda \in \rho(D)$,*

$$\begin{aligned}\mathcal{A} - \lambda &= \begin{pmatrix} I & B(D - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D - \lambda \end{pmatrix} \\ &\quad \begin{pmatrix} I & 0 \\ (D - \lambda)^{-1}C & I \end{pmatrix},\end{aligned}$$

where $S_1(\lambda) := A - \lambda - B(D - \lambda)^{-1}C$ is the first Schur complement of \mathcal{A} with domain $\mathcal{D}(S_1(\lambda)) = \mathcal{D}(A) \cap \mathcal{D}(C)$.

- (b) Suppose that A is closed with $\rho(A) \neq \emptyset$, and that $\mathcal{D}(A) \subset \mathcal{D}(C)$. Then for some (and hence for all) $\lambda \in \rho(A)$,

$$\mathcal{A} - \lambda = \begin{pmatrix} I & 0 \\ C(A - \lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} A - \lambda & 0 \\ 0 & S_2(\lambda) \end{pmatrix} \begin{pmatrix} I & (A - \lambda)^{-1}B \\ 0 & I \end{pmatrix},$$

where $S_2(\lambda) := D - \lambda - C(A - \lambda)^{-1}B$ is the second Schur complement of \mathcal{A} with domain $\mathcal{D}(S_2(\lambda)) = \mathcal{D}(B) \cap \mathcal{D}(D)$.

Proof. To prove the first equality, we denote by \mathcal{RST} the product of the three linear operators on the right side. It is easy to see that

$$\mathcal{D}(\mathcal{RST}) = \mathcal{D}(\mathcal{ST}) = (\mathcal{D}(A) \cap \mathcal{D}(C)) \times \mathcal{D}(D).$$

Hence, we have, by Theorem 2.1,

$$\mathcal{RST} = \mathcal{R} \times (\mathcal{S} \times \mathcal{T}),$$

this proved the first equality. Similarly, the second equality holds. ■

Definition 2.2. Let

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a block operator matrix on X with dense domain $\mathcal{D}_1 \times \mathcal{D}_2$. Then the block operator matrix

$$\mathcal{A}^\times := \begin{pmatrix} (A|_{\mathcal{D}_1})^* & (C|_{\mathcal{D}_1})^* \\ (B|_{\mathcal{D}_2})^* & (D|_{\mathcal{D}_2})^* \end{pmatrix}$$

is said to be the formal adjoint block operator matrix of \mathcal{A} .

Theorem 2.2. Let \mathcal{A} be a block operator matrix on X with dense domain. Then $\mathcal{A}^\times = \mathcal{A}^*$ if and only if \mathcal{A}^* has a matrix representation.

Proof. We only need to prove the “if” part. Assume \mathcal{A}^* has a matrix representation. Writing

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and $\mathcal{D}(\mathcal{A}) = \mathcal{D}_1 \times \mathcal{D}_2$. Then it follows from $(f, \mathcal{A}x) = (\mathcal{A}^\times f, x)$ for all $x \in \mathcal{D}(\mathcal{A})$ and all $f \in \mathcal{D}(\mathcal{A}^\times)$ that $\mathcal{A}^\times \subset \mathcal{A}^*$ (see also [18]). It remains to show $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A}^\times)$. Let

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*).$$

By Lemma 2.1 we have $Q_k g \in \mathcal{D}(\mathcal{A}^*)$ for $k = 1, 2$, so that

$$(2.2) \quad (Q_k g, \mathcal{A}y) = (\mathcal{A}^* Q_k g, y) \text{ for } y \in \mathcal{D}(\mathcal{A}), k = 1, 2.$$

Writing

$$\mathcal{A}^* Q_k g = \begin{pmatrix} h_{k1} \\ h_{k2} \end{pmatrix}, k = 1, 2.$$

By taking $k = 1$ in (2.2) we get

$$(g_1, A|_{\mathcal{D}_1} y_1) + (g_1, B|_{\mathcal{D}_1} y_2) = (h_{11}, y_1) + (h_{12}, y_2), \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}).$$

It follows that

$$\begin{aligned} (g_1, A|_{\mathcal{D}_1} y_1) &= (h_{11}, y_1) \text{ for } y_1 \in \mathcal{D}(A|_{\mathcal{D}_1}), \\ (g_1, B|_{\mathcal{D}_1} y_2) &= (h_{12}, y_2) \text{ for } y_2 \in \mathcal{D}(B|_{\mathcal{D}_1}), \end{aligned}$$

so that

$$(2.3) \quad g_1 \in \mathcal{D}((A|_{\mathcal{D}_1})^*) \cap \mathcal{D}((B|_{\mathcal{D}_1})^*).$$

Similarly, by taking $k = 2$ in (2.2) we get

$$(2.4) \quad g_2 \in \mathcal{D}((C|_{\mathcal{D}_1})^*) \cap \mathcal{D}((D|_{\mathcal{D}_1})^*).$$

From (2.3) and (2.4) we get $g \in \mathcal{D}(\mathcal{A}^\times)$, so that $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A}^\times)$. ■

Obviously, $\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \subset \mathcal{A}^\times \subset \mathcal{A}^*$. Moreover, there is a block operator matrix \mathcal{A} such that $\mathcal{A}^\times = \mathcal{A}^*$ but $\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \neq \mathcal{A}^*$, see the following example (for the notions of differential operators see [21]).

Example 2.1. *Given the following four differential operators on the Hilbert space $L^2(0, 1)$:*

$$\begin{aligned} \mathcal{D}(L) &:= H^2(0, 1), Lf := -f'', \\ \mathcal{D}(L_0) &:= \{f \in \mathcal{D}(L) \mid f^{(k)}(0) = f^{(k)}(1) = 0, k = 0, 1\}, L_0 := L|_{\mathcal{D}(L_0)}, \\ \mathcal{D}(L_D) &:= \{f \in \mathcal{D}(L) \mid f(0) = f(1) = 0\}, L_D := L|_{\mathcal{D}(L_D)}, \\ \mathcal{D}(L_N) &:= \{f \in \mathcal{D}(L) \mid f'(0) = f'(1) = 0\}, L_N := L|_{\mathcal{D}(L_N)}. \end{aligned}$$

It is well known that L_0 is closed, $L = L_0^$, $L_D = L_D^*$, and $L_N = L_N^*$. For the block operator matrix*

$$\mathcal{L} := \begin{pmatrix} L_D & L_N \\ L_N & -L_D \end{pmatrix}$$

defined on the Hilbert space $L^2(0, 1) \times L^2(0, 1)$, we have

- (a) \mathcal{L} is closed,
- (b) $\mathcal{L} = \begin{pmatrix} L_D^* & L_N^* \\ L_N^* & -L_D^* \end{pmatrix} \subsetneq \mathcal{L}^\times = \mathcal{L}^*$.

In fact, we see from $\mathcal{D}(L_0) = \mathcal{D}(L_D) \cap \mathcal{D}(L_N)$ that

$$\mathcal{L} = \begin{pmatrix} L_0 & L_0 \\ L_0 & -L_0 \end{pmatrix} = \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} L_0 & 0 \\ 0 & L_0 \end{pmatrix},$$

where the latter equality follows from Theorem 2.1. By Lemma A.1,

$$(2.5) \quad \mathcal{L}^* = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix},$$

which implies $\mathcal{D}(\mathcal{L}^) = \mathcal{D}(L) \times \mathcal{D}(L)$. Thus, by Theorem 2.1,*

$$\mathcal{L}^* = \begin{pmatrix} L & L \\ L & -L \end{pmatrix} = \mathcal{L}^\times.$$

But

$$\begin{pmatrix} L_D^* & L_N^* \\ L_N^* & -L_D^* \end{pmatrix} = \mathcal{L} \subsetneq \mathcal{L}^*.$$

It remains to prove \mathcal{L} is closed. Since $\frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a self-adjoint unitary operator, we have, by applying Lemma A.2 to (2.5),

$$\mathcal{L}^{**} = \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} L^* & 0 \\ 0 & L^* \end{pmatrix} = \mathcal{L}.$$

If a densely defined operator \mathcal{A} has a matrix representation, this need not be true for \mathcal{A}^* even if \mathcal{A} is closed; see the following example.

Example 2.2. Let $X_1 = X_2$ and let A be a closed densely defined operator on X_1 with $\mathcal{D}(A) \neq X_1$. Consider the block operator matrix

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix}$$

on $X_1 \times X_1$. It is easy to verify that \mathcal{A} is a closed operator with domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times X_1$. Furthermore, we see from Theorem 2.1 that

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

By Lemma A.1,

$$\mathcal{A}^* = \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix},$$

so that

$$\mathcal{D}(\mathcal{A}^*) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1 \times X_1 \mid x_1 + x_2 \in \mathcal{D}(A^*) \right\}.$$

Since A is unbounded, we have $\mathcal{D}(A^*) \neq X_1$. Taking $x_1 \in X_1 \setminus \mathcal{D}(A^*)$, then

$$\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*) \text{ but } \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \notin \mathcal{D}(\mathcal{A}^*).$$

Hence, by Lemma 2.1, \mathcal{A}^* has no block operator matrix representation.

3. SELF-ADJOINTNESS

Let H_1, H_2 be complex Hilbert spaces. Now we consider self-adjointness of block operator matrices with unbounded entries acting on the Hilbert space $H_1 \times H_2$.

First we shall consider necessary conditions for a block operator matrix to be self-adjoint. Let $\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block operator matrix acting on $H_1 \times H_2$ with dense domain $\mathcal{D}_1 \times \mathcal{D}_2$. Clearly, \mathcal{A} is symmetric if and only if

$$(3.1) \quad A|_{\mathcal{D}_1} \subset (A|_{\mathcal{D}_1})^*, B|_{\mathcal{D}_2} \subset (C|_{\mathcal{D}_1})^*, D|_{\mathcal{D}_2} \subset (D|_{\mathcal{D}_2})^*.$$

Furthermore, it follows from Theorem 2.2 and (3.1) the following assertion.

Proposition 3.1. *If \mathcal{A} is self-adjoint, then*

$$\mathcal{A} = \begin{pmatrix} \overline{A|_{\mathcal{D}_1}} & \overline{B|_{\mathcal{D}_2}} \\ \overline{C|_{\mathcal{D}_1}} & \overline{D|_{\mathcal{D}_2}} \end{pmatrix} = \mathcal{A}^\times.$$

In addition, we point out that the entry operators of a self-adjoint block operator matrix need not be closed, see the following example.

Example 3.1. Let $X := (L^2(0, 1) \times L^2(0, 1)) \times (L^2(0, 1) \times L^2(0, 1))$. Let M be the differential operator on the Hilbert space $L^2(0, 1)$ which is defined by

$$\mathcal{D}(M) := \{f \in H^1(0, 1) \mid f(0) = f(1) = 0\}, Mf := if'.$$

From the methods of differential operators we know that M is closed, $C_c^\infty(0, 1)$ is a core of M , and M^* is determined by

$$\mathcal{D}(M^*) := H^1(0, 1), M^*f := if'.$$

Let L_D be the same as in Example 2.1 and let $M_0 := M|_{\mathcal{D}(L_D)}$. For the block operator matrix

$$\mathcal{A} := \begin{pmatrix} L_D & 0 & M_0 & 0 \\ 0 & M_0 & 0 & L_D \\ M_0 & 0 & L_D & 0 \\ 0 & L_D & 0 & M_0 \end{pmatrix} =: \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

on the Hilbert space X , we claim that

- (a) \mathcal{A} is self-adjoint,
- (b) A, B are not closed and $A \subsetneq A^*, B \subsetneq B^*$.

In fact, it is easy to see that

$$\mathcal{A} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} L_D & M_0 & 0 & 0 \\ M_0 & L_D & 0 & 0 \\ 0 & 0 & M_0 & L_D \\ 0 & 0 & L_D & M_0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} =: \mathcal{E}\mathcal{B}\mathcal{E}.$$

By interpolation theorem of Sobolev spaces (see [1, Theorem 5.2]) and Lemma A.3, the operators $\begin{pmatrix} M_0 & L_D \\ L_D & M_0 \end{pmatrix}$ and $\begin{pmatrix} L_D & M_0 \\ M_0 & L_D \end{pmatrix}$ are self-adjoint. Hence, \mathcal{B} is self-adjoint (see [29, Proposition 2.6.3]). Since $\mathcal{E}^* = \mathcal{E}^{-1} = \mathcal{E}$, it follows from Lemma A.2 and Lemma A.3 that \mathcal{A} is self-adjoint. This proved the first claim. The second claim follows from the following four equalities:

$$\begin{aligned} A^* &= \begin{pmatrix} L_D & 0 \\ 0 & M^* \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} L_D & 0 \\ 0 & M \end{pmatrix}, \\ B^* &= \begin{pmatrix} M^* & 0 \\ 0 & L_D \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} M & 0 \\ 0 & L_D \end{pmatrix}. \end{aligned}$$

Next we consider sufficient conditions for a block operator matrix to be (essentially) self-adjoint. In view of (3.1) and Example 3.1, through out the rest of this section we make the following *basic assumptions*:

- (i) A, B, C, D are densely defined and closable,
- (ii) $A \subset A^*, B \subset C^*, D \subset D^*$,
- (iii) $\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is densely defined on $H_1 \times H_2$.

Further assumptions will be formulated where they are needed.

Proposition 3.2. \mathcal{A} is self-adjoint if one of the following statements holds:

- (a) A, D are self-adjoint, C is A -bounded with relative bound < 1 , and B is D -bounded with relative bound < 1 .
- (b) B is closed, $C = B^*$, A is C -bounded with relative bound < 1 , and D is B -bounded with relative bound < 1 .

Proof. We prove the claim in case (a); the proof in case (b) is analogous. Writing

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} =: \mathcal{S} + \mathcal{T}.$$

Then \mathcal{S} is self-adjoint and \mathcal{T} is symmetric (see [29, Proposition 2.6.3]). Furthermore, by the assumptions \mathcal{T} is \mathcal{S} -bounded with relative bound < 1 . By applying Lemma A.3 to \mathcal{S}, \mathcal{T} , we complete the proof. ■

Proposition 3.3. *\mathcal{A} is essentially self-adjoint if one of the following statements holds:*

(a) *A, D are self-adjoint, and for some $a \geq 0$,*

$$\|Cx\|^2 \leq a\|x\|^2 + \|Ax\|^2, x \in \mathcal{D}(A),$$

$$\|By\|^2 \leq a\|y\|^2 + \|Dy\|^2, y \in \mathcal{D}(D).$$

(b) *B is closed, $C = B^*$, and for some $a \geq 0$,*

$$\|Ax\|^2 \leq a\|x\|^2 + \|Cx\|^2, x \in \mathcal{D}(C),$$

$$\|Dy\|^2 \leq a\|y\|^2 + \|By\|^2, y \in \mathcal{D}(B).$$

Proof. We prove e.g. case (a). By the assumptions, we have, for all $(x, y)^t \in \mathcal{D}(A) \times \mathcal{D}(D)$,

$$\left\| \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq \sqrt{a} \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| + \left\| \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|.$$

Consequently, the assertion follows from the Wüst theorem (see [30, Theorem 4]). ■

Theorem 3.1. *The following statements hold.*

(a) *If $D = D^*$, $\mathcal{D}(D) \subset \mathcal{D}(B)$, then \mathcal{A} is self-adjoint if and only if*

$$(A - B(D - \lambda)^{-1}C)^* = A - B(D - \bar{\lambda})^{-1}C$$

for some (and hence for all) $\lambda \in \rho(D)$.

(b) *If $A = A^*$, $\mathcal{D}(A) \subset \mathcal{D}(C)$, then \mathcal{A} is self-adjoint if and only if*

$$(D - C(A - \lambda)^{-1}B)^* = D - C(A - \bar{\lambda})^{-1}B$$

for some (and hence for all) $\lambda \in \rho(A)$.

Proof. Proof of (a). Let $\lambda \in \rho(D)$. By applying Corollary 2.1 to $(\mathcal{A} - \lambda)$ we have

$$(3.2) \quad \mathcal{A} - \lambda = \begin{pmatrix} I & B(D - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D - \lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (D - \lambda)^{-1}C & I \end{pmatrix},$$

where $S_1(\lambda) := A - \lambda - B(D - \lambda)^{-1}C$. We see from $\mathcal{D}(D) \subset \mathcal{D}(B), B \subset C^*$ that $\mathcal{D}(D) \subset \mathcal{D}(C^*)$, so that $(D - \lambda)^{-1}C$ is bounded on its domain $\mathcal{D}(C)$ (see [3, Proposition 3.1]). Moreover, in (3.2), the domain of the middle factor is equal to $(\mathcal{D}(A) \cap \mathcal{D}(C)) \times \mathcal{D}(D)$, and so we can replace $(D - \lambda)^{-1}C = \overline{((D - \lambda)^{-1}C)|_{\mathcal{D}(C)}}$ by

$$\begin{aligned} \overline{(D - \lambda)^{-1}C} &= ((D - \lambda)^{-1}C)^{**} \\ &= (C^*(D - \bar{\lambda})^{-1})^* \quad (\text{by Lemma A.1}) \\ &= (B(D - \bar{\lambda})^{-1})^*. \end{aligned}$$

It follows that

$$(3.3) \quad \mathcal{A} - \lambda = \begin{pmatrix} I & B(D - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D - \lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (B(D - \bar{\lambda})^{-1})^* & I \end{pmatrix}.$$

In the factorization (3.3), the first and last factor are bounded and boundedly invertible, and therefore by Lemma A.1 and Lemma A.2,

$$(3.4) \quad \mathcal{A}^* - \bar{\lambda} = \begin{pmatrix} I & B(D - \bar{\lambda})^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\bar{\lambda})^* & 0 \\ 0 & D - \bar{\lambda} \end{pmatrix} \begin{pmatrix} I & 0 \\ (B(D - \lambda)^{-1})^* & I \end{pmatrix}.$$

Furthermore, in (3.3) we can replace λ by $\bar{\lambda}$ and obtain

$$(3.5) \quad \mathcal{A} - \bar{\lambda} = \begin{pmatrix} I & B(D - \bar{\lambda})^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\bar{\lambda}) & 0 \\ 0 & D - \bar{\lambda} \end{pmatrix} \begin{pmatrix} I & 0 \\ (B(D - \lambda)^{-1})^* & I \end{pmatrix}.$$

We conclude from (3.4),(3.5) that $\mathcal{A}^* = \mathcal{A}$ if and only if $S_1(\lambda)^* = S_1(\bar{\lambda})$.

Proof of (b). Let $\lambda \in \rho(A)$. Similar to the proof of (a) we have $\mathcal{A}^* = \mathcal{A}$ if and only if $S_2(\lambda)^* = S_2(\bar{\lambda})$, where $S_2(\lambda) := D - \lambda - C(A - \lambda)^{-1}B$. ■

Corollary 3.1. *Let $A = A^*, D = D^*$. Then \mathcal{A} is self-adjoint if one of the following holds:*

- (a) *C is A -bounded with relative bound < 1 and B is D -bounded with relative bound ≤ 1 ,*
- (b) *C is A -bounded with relative bound ≤ 1 and B is D -bounded with relative bound < 1 .*

Proof. We prove the claim in case (a); the proof in case (b) is analogous. It is enough to prove

$$(3.6) \quad (A - B(D - i\lambda)^{-1}C)^* = A - B(D + i\lambda)^{-1}C \text{ for some } \lambda > 0.$$

Step 1. We start from the claim that for $\lambda > 0$ large enough, $B(D - i\lambda)^{-1}C$ is A -bounded with relative bound < 1 . Since C is A -bounded with relative bound < 1 , it is enough to prove for each $\varepsilon > 0$ there exists $\lambda > 0$ such that

$$\|B(D - i\lambda)^{-1}\| < 1 + \varepsilon.$$

We observe that for $x \in \mathcal{D}(D)$ and $\lambda > 0$,

$$(3.7) \quad \|(D - i\lambda)x\|^2 = \|Dx\|^2 + \lambda^2\|x\|^2$$

since D is self-adjoint. By the assumption that B is D -bounded with relative bound ≤ 1 , there exists $a(\varepsilon) \geq 0$ such that

$$(3.8) \quad \|Bx\| \leq (1 + \frac{\varepsilon}{2})\|Dx\| + a(\varepsilon)\|x\|, x \in \mathcal{D}(D),$$

so that for $x \in \mathcal{D}(D)$, we have, using (3.7) twice and then (3.8),

$$\|Bx\| \leq (1 + \frac{\varepsilon}{2} + \frac{a(\varepsilon)}{\lambda})\|(D - i\lambda)x\|.$$

It is enough to choose $\lambda > 0$ large enough such that $\frac{a(\varepsilon)}{\lambda} < \frac{\varepsilon}{2}$.

Step 2. In this step, we show that for $\lambda > 0$ large enough, $(B(D - i\lambda)^{-1}C)^*$ is A -bounded with relative bound < 1 . Since $\mathcal{D}(D) \subset \mathcal{D}(B)$ and B is closable, $B(D - i\lambda)^{-1}$ is everywhere defined and closed, so that it is bounded by the closed graph theorem. Thus, by Lemma A.1,

$$(B(D - i\lambda)^{-1}C)^* = C^*(B(D - i\lambda)^{-1})^* = C^*\overline{(D + i\lambda)^{-1}B^*}.$$

Now $(D + i\lambda)^{-1}B^*$ is bounded on $\mathcal{D}(B^*)$ since $\mathcal{D}(B) \supset \mathcal{D}(D)$ (see [3, Proposition 3.1]), so that $(C^*\overline{(D + i\lambda)^{-1}B^*})|_{\mathcal{D}(B^*)} = C^*(D + i\lambda)^{-1}B^* = B(D + i\lambda)^{-1}B^*$. Then it follows from $\mathcal{D}(A) \subset \mathcal{D}(C)$, $C \subset B^*$ that

$$(3.9) \quad (B(D - i\lambda)^{-1}C)^*|_{\mathcal{D}(A)} = (B(D + i\lambda)^{-1}C)|_{\mathcal{D}(A)}.$$

Thus, by *Step 1*, $(B(D - i\lambda)^{-1}C)^*$ is A -bounded with relative bound < 1 .

Step 3. Now (3.6) follows from *Step 1* and *Step 2* by applying Lemma A.3 and (3.9). ■

Corollary 3.2. *Let $A = A^*$, $D = D^*$. Then \mathcal{A} is self-adjoint if one of the following holds:*

- (a) C is A -bounded with relative bound 0, and $\mathcal{D}(D) \subset \mathcal{D}(B)$.
- (b) $\mathcal{D}(A) \subset \mathcal{D}(C)$, and B is D -bounded with relative bound 0.

Proof. We prove the claim in case (a); the proof in case (b) is analogous. Let $\lambda \in \rho(A)$. We need to prove

$$(3.10) \quad (D - C(A - \lambda)^{-1}B)^* = D - C(A - \bar{\lambda})^{-1}B.$$

Step 1. First we claim that $C(A - \lambda)^{-1}B$ is D -bounded with relative bound 0. Since C is A -bounded with relative bound 0, for each $\varepsilon > 0$, there exists $b_1(\varepsilon, \lambda) \geq 0$, such that

$$\|C(A - \lambda)^{-1}x\| \leq \varepsilon\|x\| + b_1(\varepsilon, \lambda)\|(A - \lambda)^{-1}x\|, x \in H_1,$$

so that for $x \in \mathcal{D}(B)$,

$$\begin{aligned} \|C(A - \lambda)^{-1}Bx\| &\leq \varepsilon\|Bx\| + b_1(\varepsilon, \lambda)\|(A - \lambda)^{-1}Bx\| \\ &\leq \varepsilon\|Bx\| + b_2(\varepsilon, \lambda)\|x\|, \end{aligned}$$

where the last inequality follows from the fact that $(A - \lambda)^{-1}B$ is bounded on $\mathcal{D}(B)$ (since $\mathcal{D}(B^*) \supset \mathcal{D}(C) \supset \mathcal{D}(A)$). Furthermore, since D is closed and $\mathcal{D}(D) \subset \mathcal{D}(B)$, there are $a, b \geq 0$ such that

$$\|Bx\| \leq a\|Dx\| + b\|x\|, x \in \mathcal{D}(D),$$

so that

$$\|C(A - \lambda)^{-1}Bx\| \leq \varepsilon a\|Dx\| + b_3(\varepsilon, \lambda)\|x\|, x \in \mathcal{D}(D).$$

Step 2. We have, with arguments similar to the ones used in the proof of (3.9),

$$(C(A - \lambda)^{-1}B)^*|_{\mathcal{D}(D)} = C(A - \bar{\lambda})^{-1}B|_{\mathcal{D}(D)},$$

so that by *Step 1*, $(C(A - \lambda)^{-1}B)^*$ is D -bounded with relative bound 0.

Step 3. Finally, (3.10) follows from *Step 1* and *Step 2* by applying Lemma A.3. ■

The following analogue of Theorem 3.1 can be proved in the same way.

Theorem 3.2. *The following statements hold.*

- (a) If $D = D^*$ with $\mathcal{D}(D) \subset \mathcal{D}(B)$, then \mathcal{A} is essentially self-adjoint if and only if

$$(A - B(D - \lambda)^{-1}C)^* = \overline{A - B(D - \bar{\lambda})^{-1}C}$$

for some (and hence for all) $\lambda \in \rho(D)$.

- (b) If $A = A^*$ with $\mathcal{D}(A) \subset \mathcal{D}(B^*)$, then \mathcal{A} is essentially self-adjoint if and only if

$$(D - C(A - \lambda)^{-1}B)^* = \overline{D - C(A - \bar{\lambda})^{-1}B}$$

for some (and hence for all) $\lambda \in \rho(A)$.

Corollary 3.3. *The following statements hold.*

- (a) Let A be self-adjoint with $\mathcal{D}(|A|^{\frac{1}{2}}) \subset \mathcal{D}(C)$ and let D be essentially self-adjoint. If $\mathcal{D}(B) \cap \mathcal{D}(D)$ is a core of \overline{D} , then \mathcal{A} is essentially self-adjoint.
(b) Let A be essentially self-adjoint and let D be self-adjoint with $\mathcal{D}(|D|^{\frac{1}{2}}) \subset \mathcal{D}(B)$. If $\mathcal{D}(A) \cap \mathcal{D}(C)$ is a core of \overline{A} , then \mathcal{A} is essentially self-adjoint.

Proof. We prove the claim in case (a); the proof in case (b) is analogous. Let $\lambda \in \rho(A)$, we start to prove

$$(D - C(A - \lambda)^{-1}B)^* = \overline{D - C(A - \bar{\lambda})^{-1}B}$$

via the arguments used in [17, Section 1]. Since $C \subset B^*$ and $\mathcal{D}(|A|^{\frac{1}{2}}) \subset \mathcal{D}(C)$, the operator

$$C(|A| + I)^{-\frac{1}{2}} = B^*(|A| + I)^{-\frac{1}{2}}$$

is closed and defined on the whole space. It follows from the closed graph theorem that $C(|A| + I)^{-\frac{1}{2}}$ is bounded. Consequently, the operator

$$(|A| + I)^{-\frac{1}{2}}B = (B^*(|A| + I)^{-\frac{1}{2}})^*|_{\mathcal{D}(B)}$$

is bounded on $\mathcal{D}(B)$. Since the operator

$$U(\lambda) := (|A| + I)^{\frac{1}{2}}(A - \lambda)^{-1}(|A| + I)^{\frac{1}{2}}$$

is bounded on its domain $\mathcal{D}(|A|^{\frac{1}{2}})$, the operator

$$C(A - \lambda)^{-1}B = C(|A| + I)^{-\frac{1}{2}}U(\lambda)(|A| + I)^{-\frac{1}{2}}B$$

is also bounded on its domain $\mathcal{D}(B)$. Hence, if $\mathcal{D}(B) \cap \mathcal{D}(D)$ is a core of \overline{D} , then by Lemma A.3,

$$\begin{aligned} (D - C(A - \lambda)^{-1}B)^* &= D^* - (C(A - \lambda)^{-1}B)^* \\ &= \overline{D - C(A - \bar{\lambda})^{-1}B} \\ &= \overline{D - C(A - \bar{\lambda})^{-1}B}. \end{aligned}$$

■

Remark 3.1. *It follows from Corollary 3.3 and Theorem 3.1 that the linearized Navier-Stokes operator considered in [6] is essentially self-adjoint and it is not closed.*

By the techniques used in the proofs of [2, Theorem 3.1] and the corresponding corollaries therein, with some slight modifications, we can prove the following theorem and related corollaries.

Theorem 3.3. *Let $H_1 = H_2$ and let $C = B^*$. If B is closed with $\rho(B) \neq \emptyset$ and $\mathcal{D}(\mathcal{A}) = \mathcal{D}(B^*) \times \mathcal{D}(B)$, then the following statements are equivalent:*

- (a) \mathcal{A} is self-adjoint,
- (b) $(B - A(B^* - \overline{\lambda})^{-1}D)^* = B^* - D(B - \lambda)^{-1}A$ for some (and hence for all) $\lambda \in \rho(B)$,
- (c) $(B^* - D(B - \lambda)^{-1}A)^* = B - A(B^* - \overline{\lambda})^{-1}D$ for some (and hence for all) $\lambda \in \rho(B)$.

Corollary 3.4. *Let $H_1 = H_2$ and let $B = B^* = C$. Then \mathcal{A} is self-adjoint if one of the following holds:*

- (a) A is B -bounded with relative bound < 1 and D is B -bounded with relative bound ≤ 1 ,
- (b) A is B -bounded with relative bound ≤ 1 and D is B -bounded with relative bound < 1 .

For the definition and properties of a maximal accretive operator in the following corollary, see [20, Section IV.4].

Corollary 3.5. *Let $H_1 = H_2$ and let $C = B^*$. If B or $-B$ is maximal accretive, then \mathcal{A} is self-adjoint if one of the following holds:*

- (a) A is B^* -bounded with relative bound < 1 and D is B -bounded with relative bound ≤ 1 ,
- (b) A is B^* -bounded with relative bound ≤ 1 and D is B -bounded with relative bound < 1 .

Corollary 3.6. *Let $H_1 = H_2$ and let $C = B^*$. If B is closed with $\rho(B) \neq \emptyset$ and $\mathcal{D}(\mathcal{A}) = \mathcal{D}(B^*) \times \mathcal{D}(B)$, then \mathcal{A} is self-adjoint if one of the following holds:*

- (a) A is B^* -bounded with relative bound 0,
- (b) D is B -bounded with relative bound 0.

APPENDIX A. SOME LEMMAS ON ADJOINTS

In this section, we will denote by X, Y, Z Banach spaces.

Let S, T be linear operators from X to Y and X to Z , respectively. Recall that S is called T -bounded if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist constants $a, b \geq 0$ such that

$$\|Sx\| \leq a\|x\| + b\|Tx\|, x \in \mathcal{D}(T),$$

see [9, Section VI.1.1]. The greatest lower bound b_0 of all possible constants b in the above inequality will be called the relative bound of S with respect to T (or simply the relative bound when there is no confusion). If T is closed and S is closable, then $\mathcal{D}(T) \subset \mathcal{D}(S)$ already implies that S is T -bounded (see [9, Remark IV.1.5]).

Lemma A.1. ([9, Problem III.5.26]) *Let S be a bounded everywhere defined operator from Y to Z and let T be a densely defined operator from X to Y . Then $(ST)^* = T^*S^*$.*

Lemma A.2. ([24]) *Let S be a densely defined operator from Y to Z and let T be a closed densely defined operator from X to Y . If the range $\mathcal{R}(T)$ of T is closed in Y and has finite codimension, then ST is a densely defined operator and $(ST)^* = T^*S^*$.*

Lemma A.3. ([8, Corollary 1]) *Let T be closed densely defined operators from X to Y . Suppose S is a T -bounded operator such that S^* is T^* -bounded, with both relative bounds < 1 . Then $S + T$ is closed and $(S + T)^* = S^* + T^*$.*

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